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Maximal symmetry group of the Hamilton–Jacobi equation: relativistic particle in flat spacetime

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Abstract. Lie's extended group method has been used to obtain the maximal symmetry group of the Hamilton-Jacobi equation for a relativistic particle moving in flat spacetime. For a massive particle it is the 21-parameter inhomogeneous pseudo-orthogonal group IO(4, 1) of signature (4, 1). For a zero-mass particle like a photon or a neutrino this group is an infinite-parameter Lie group with an infinite-parameter invariant subgroup such that the factor group is isomorphic to the inhomogeneous pseudo-orthogonal group IO(3, 1) of signature (3, 1).

1. Introduction

From the point of view of physics a problem is completely solved if the solutions of the corresponding dynamical equation of motion are known. In obtaining these solutions and classifying them, the knowledge of the full symmetry group of the dynamical equation of motion is, perhaps, of the greatest help (McIntosh 1971). In a series of papers Leach and Prince (Prince and Leach 1980, Prince and Eliezer 1980, 1981, Leach 1981, Prince 1983) solved the classical problem of Kepler motion and that of the N-dimensional harmonic oscillator. The one-dimensional classical harmonic oscillator problem was also discussed by Lutzky (1978) and Wulfman and Wybourne (1976). In all these problems the Lagrangian equation or the Hamilton canonical equation was considered and Lie's extended group method (Sattinger 1977, Hamermesh 1984, Rudra 1984) was used to obtain the maximal symmetry group of the corresponding dynamical equation of motion.

Here we apply Lie's method to obtain the maximal symmetry group of the dynamical equation of motion for a relativistic particle in flat spacetime. For the relativistic problem the Hamilton-Jacobi equation is the most suitable dynamical equation of motion, since the time and the space coordinates are treated on an equal footing. Thus we have considered this equation of motion. Our analysis shows that, for a massive particle, the maximal symmetry group is the 21-parameter inhomogeneous pseudo-orthogonal group IO(4, 1) that keeps the equation $\Sigma_{i=1}^{4}(\mathrm{d}q^i)^2-(\mathrm{d}q^5)^2=0$ invariant. For zero-mass particles, like the photon and the neutrino, the maximal symmetry group is an infinite-parameter Lie group with an infinite-parameter invariant subgroup such that the factor group is isomorphic to the inhomogeneous pseudo-orthogonal group IO(3, 1) that keeps the equation $\Sigma_{i=1}^{3}(\mathrm{d}q^i)^2-(\mathrm{d}q^4)^2=0$ invariant.

In § 2 we give a brief summary of Lie's method for obtaining the maximal symmetry group of a differential equation. In § 3 we apply this method to the Hamilton-Jacobi equation for a relativistic particle in flat spacetime.

2. Lie's extended group method

In this section we summarise Lie's extended group method. By maximal symmetry group we mean the group G with generators X for transformations in the space of n independent variables q^i, $i=1,\dots,n$ and s dependent variables Ψ^k, $k=1,\dots,s$,

$$X=\sum_i \xi^i(q,\Psi)\,\partial/\partial q^i+\sum_k \varphi_k(q,\Psi)\,\partial/\partial\Psi^k, \tag{1}$$

where ξ^i and φ_k are the vectors of the generators (Eisenhart 1961). These transformations are such that the forms of the set of partial differential equations

$$\Delta^\alpha(q,\Psi;r)=0, \qquad \alpha=1,\dots,p, \tag{2}$$

where r denotes the highest order of partial derivatives of Ψ, are kept invariant. The algorithm for obtaining the vectors of the generators is the following. We construct the rth extension of X

$$X^{(r)}=X+\sum_k \sum_{1\leqslant|J|\leqslant r} \varphi_k^J(q,\Psi,\Psi_J)\,\partial/\partial\Psi_J^k. \tag{3}$$

Here

$$J\equiv(j_1,\dots,j_n), \qquad |J|=\sum_i j_i, \qquad \Psi_J^k=\partial^{|J|}\Psi^k/\prod_i(\partial q^i)^{j_i},$$

$$\varphi_k^J=D^J\Big(\varphi_k-\sum_i \Psi_i^k\xi^i\Big)+\sum_i \Psi_{J,i}^k\xi^i,$$

$$D^J=\prod_i D_i^{j_i}, \qquad D_i=\partial/\partial q^i+\sum_k \sum_{0\leqslant|J|\leqslant r} \Psi_{J,i}^k\,\partial/\partial\Psi_J^k,$$

$$\Psi_i^k=\partial\Psi^k/\partial q^i, \qquad (J,i)\equiv(j_1,\dots,j_{i-1},j_i+1,j_{i+1},\dots,j_n). \tag{4}$$

In all these expressions, while taking the partial derivatives, q^i, Ψ^k and Ψ_j^k are to be considered as independent variables.

X is a generator of G if

$$X^{(r)}\,\Delta^\alpha=0, \qquad \alpha=1,\dots,p. \tag{5}$$

We use equation (2) in the left-hand side of equation (5) and obtain the coefficients of different powers and products of the different orders of partial derivatives of Ψ^k. When we separately equate these coefficients to zero, we get a set of partial differential equations for ξ and φ. The solutions of these partial differential equations give us the most general form of X and hence the maximal symmetry group G.

3. The Hamilton–Jacobi equation in flat spacetime

For a particle of mass m in flat spacetime the Hamilton–Jacobi equation (Landau and Lifshitz 1962) is

$$S_\tau^2-\sum_\alpha S_\alpha^2-(mc)^2=0.$$

Here the independent coordinates are q^α $(\alpha=1,2,3)$ and $\tau=ct$, the action S is the dependent variable and c is the speed of light in vacuum. The subscript Greek letters α and τ denote partial derivatives with respect to q^α and τ. If we divide S by mc and call the resulting quantity the new normalised action (retaining the same symbol S), then for this normalised action the differential equation is

$$\Delta \equiv S_\tau^2 - \sum_\alpha S_\alpha^2 - 1 = 0. \tag{6}$$

The generator for the maximal symmetry group for equation (6) is written as

$$X = \xi^\tau X^\tau + \sum_\alpha \xi^\alpha X^\alpha + \varphi X^S,$$

where

$$X^\tau = -\mathrm{i}\,\partial/\partial\tau, \qquad X^\alpha = -\mathrm{i}\,\partial/\partial q^\alpha, \qquad X^S = -\mathrm{i}\,\partial/\partial S. \tag{7}$$

Equation (5) will give us the following partial differential equations for ξ and φ:

$$\xi^\alpha_\alpha = \xi^\tau_\tau, \qquad \xi^\beta_\alpha + \xi^\alpha_\beta = 0, \qquad \xi^\tau_\alpha = \xi^\alpha_\tau, \qquad \forall \alpha, \beta, \alpha \neq \beta, \tag{8a}$$

$$\varphi_\tau = \xi^\tau_S, \qquad \varphi_\alpha = -\xi^\alpha_S, \qquad \varphi_S = \xi^\tau_\tau, \qquad \forall \alpha. \tag{8b}$$

From (8a) we find that all third-order partial derivatives of ξ^τ and ξ^α vanish:

$$\partial^3 \xi^{\tau,\alpha}/(\partial\tau)^{n_4} \prod_\alpha (\partial q^\alpha)^{n_\alpha} = 0, \qquad n_4 + \sum_\alpha n_\alpha = 3. \tag{9}$$

Thus the general form of ξ consistent with (8a) and (9) is

$$\xi^\tau = a_4(S) + \sum_\beta b_{3+\beta}(S) q^\beta + b(S)\tau + C(S)(r^2+\tau^2)/2 + \tau \sum_\beta C_\beta(S) q^\beta,$$

$$\xi^\alpha = a_\alpha(S) + b(S) q^\alpha + \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_\gamma(S) q^\beta + b_{3+\alpha}(S)\tau - C_\alpha(S)(r^2-\tau^2)/2$$
$$+ q^\alpha \sum_\beta C_\beta(S) q^\beta + C(S) q^\alpha \tau, \qquad r^2 = \sum_\alpha (q^\alpha)^2. \tag{10}$$

We also find that the second partial derivatives of ξ with respect to q^α and τ are independent of S. Thus $C_\alpha(S)$ and $C(S)$ are constants

$$C_\alpha(S) = C_\alpha, \qquad C(S) = C. \tag{11}$$

We now expand $\varphi(q, \tau, S)$ in power series of S with coefficients as functions of q^α and τ:

$$\varphi = \sum_{n=0}^{\infty} d_n(q, \tau) S^n/n! \tag{12}$$

From (8b) we see that $\varphi_{\alpha SS} = \varphi_{\tau SS} = 0$. Thus $d_n(q, \tau)$ for $n \geq 2$ are independent of q^α and τ:

$$\varphi = d_0(q, \tau) + d_1(q, \tau) S + \sum_{n=2}^{\infty} d_n S^n/n!. \tag{13}$$

We now note that $\varphi_{\alpha\tau} = 0$, i.e. $d_0(q, \tau)$ and $d_1(q, \tau)$ are of the form

$$d_n(q, \tau) = d_n^1(q) + d_n^2(\tau), \qquad n = 0, 1. \tag{14}$$

Using the relations $\varphi_{\tau\tau} = -\varphi_{\alpha\alpha} = \varphi_{SS}$ we obtain

$$\varphi = \left(d_0 + \sum_\alpha d_0^\alpha q^\alpha + d_0^4 \tau - d_2(r^2 - \tau^2) \right)$$
$$+ \left(d_1 + \sum_\alpha d_1^\alpha q^\alpha + d_1^4 \tau - d_3(r^2 - \tau^2) \right) S + d_2 S^2/2! + d_3 S^3/3!. \tag{15}$$

Further use of (8a) and (8b) finally gives us

$$\xi^{\tau}=(a_4^0+d_0^4S+C^0S^2/2)+(d_1+d_2S)\tau+\sum_{\beta}b_{3+\beta}^0q^{\beta}$$
$$+C^0(r^2+\tau^2)/2+\tau\sum_{\beta}C_{\beta}^0q^{\beta},$$
$$\xi^{\alpha}=-(a_{\alpha}^0+d_0^{\alpha}S+C_{\alpha}^0S^2/2)+(d_1+d_2S)q^{\alpha}+\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_{\gamma}^0q^{\beta}+b_{3+\alpha}^0\tau \tag{16}$$
$$-C_{\alpha}^0(r^2-\tau^2)/2+q^{\alpha}\sum_{\beta}C_{\beta}^0q^{\beta}+C^0q^{\alpha}\tau,$$
$$\varphi=\left(d_0+\sum_{\beta}d_0^{\beta}q^{\beta}+d_0^4\tau-d_2(r^2-\tau^2)/2\right)+\left(d_1+\sum_{\beta}C_{\beta}^0q^{\beta}+C^0\tau\right)S+\tfrac{1}{2}d_2S^2.$$

The maximal symmetry group G of the Hamilton-Jacobi equation (6) then has the following 21 generators:

$$\begin{aligned}
&X^{\alpha}=-\mathrm{i}\,\partial/\partial q^{\alpha}, && X^{\tau}=-\mathrm{i}\,\partial/\partial\tau,\\
&X^{S}=-\mathrm{i}\,\partial/\partial S, && X_0=SX^S+\tau X^{\tau}+\sum_{\alpha}q^{\alpha}X^{\alpha},\\
&X_{\mathrm{R}}^{\alpha}=\sum_{\beta\gamma}e_{\alpha\beta\gamma}q^{\beta}X^{\gamma}, && X_{\mathrm{L}}^{\alpha}=q^{\alpha}X^{\tau}+\tau X^{\alpha},\\
&X_{\mathrm{A}}^{\alpha}=-\tfrac{1}{2}(r^2-\tau^2+S^2)X^{\alpha}+q^{\alpha}X_0,\\
&X_{\mathrm{A}}^{\tau}=\tfrac{1}{2}(r^2-\tau^2+S^2)X^{\tau}+\tau X_0,\\
&X_{\mathrm{A}}^{S}=-\tfrac{1}{2}(r^2-\tau^2+S^2)X^{S}+SX_0,\\
&X_{\mathrm{S}}^{\alpha}=q^{\alpha}X^S-SX^{\alpha}, && X_{\mathrm{S}}^{\tau}=\tau X^S+SX^{\tau},
\end{aligned} \tag{17}$$

with non-vanishing commutators

$$\begin{aligned}
&[X^{\alpha},X_0]=-\mathrm{i}X^{\alpha},\ [X^{\alpha},X_{\mathrm{R}}^{\beta}]=\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X^{\gamma}, && [X^{\alpha},X_{\mathrm{L}}^{\beta}]=-\mathrm{i}\,\delta_{\alpha\beta}\,X^{\tau},\\
&[X^{\alpha},X_{\mathrm{A}}^{\beta}]=-\mathrm{i}\,\delta_{\alpha\beta}\,X_0+\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma}, && [X^{\alpha},X_{\mathrm{A}}^{\tau}]=-\mathrm{i}X_{\mathrm{L}}^{\alpha},\\
&[X^{\alpha},X_{\mathrm{S}}^{\beta}]=-\mathrm{i}\,\delta_{\alpha\beta}X^{S}, && [X^{\alpha},X_{\mathrm{A}}^{S}]=\mathrm{i}X_{\mathrm{S}}^{\alpha},\\
&[X^{\tau},X_0]=-\mathrm{i}X^{\tau},\quad [X^{\tau},X_{\mathrm{L}}^{\alpha}]=-\mathrm{i}X^{\alpha}, && [X^{\tau},X_{\mathrm{A}}^{\alpha}]=-\mathrm{i}X_{\mathrm{L}}^{\alpha},\\
&[X^{\tau},X_{\mathrm{A}}^{\tau}]=-\mathrm{i}X_0,\quad [X^{\tau},X_{\mathrm{S}}^{\tau}]=-\mathrm{i}X^{S}, && [X^{\tau},X_{\mathrm{A}}^{S}]=-\mathrm{i}X_{\mathrm{S}}^{\tau},\\
&[X^{S},X_0]=-\mathrm{i}X^{S},\quad [X^{S},X_{\mathrm{A}}^{\alpha}]=-\mathrm{i}X_{\mathrm{S}}^{\alpha}, && [X^{S},X_{\mathrm{A}}^{\tau}]=-\mathrm{i}X_{\mathrm{S}}^{\tau},\\
&[X^{S},X_{\mathrm{S}}^{\alpha}]=\mathrm{i}X^{\alpha},\quad [X^{S},X_{\mathrm{S}}^{\tau}]=-\mathrm{i}X^{\tau}, && [X^{S},X_{\mathrm{A}}^{S}]=-\mathrm{i}X_0,\\
&[X_0,X_{\mathrm{A}}^{\alpha}]=-\mathrm{i}X_{\mathrm{A}}^{\alpha},\quad [X_0,X_{\mathrm{A}}^{\tau}]=-\mathrm{i}X_{\mathrm{A}}^{\tau}, && [X_0,X_{\mathrm{A}}^{S}]=-\mathrm{i}X_{\mathrm{A}}^{S},\\
&[X_{\mathrm{R}}^{\alpha},X_{\mathrm{B}}^{\beta}]=\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma}, && [X_{\mathrm{R}}^{\alpha},X_{\mathrm{L}}^{\beta}]=\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{L}}^{\gamma},\\
&[X_{\mathrm{R}}^{\alpha},X_{\mathrm{A}}^{\beta}]=\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{A}}^{\gamma}, && [X_{\mathrm{R}}^{\alpha},X_{\mathrm{S}}^{\beta}]=\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{S}}^{\gamma},\\
&[X_{\mathrm{L}}^{\alpha},X_{\mathrm{L}}^{\beta}]=-\mathrm{i}\sum_{\gamma}e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma}, && [X_{\mathrm{L}}^{\alpha},X_{\mathrm{A}}^{\beta}]=-\mathrm{i}\,\delta_{\alpha\beta}\,X_{\mathrm{A}}^{\tau},\\
&[X_{\mathrm{L}}^{\alpha},X_{\mathrm{A}}^{\tau}]=-\mathrm{i}X_{\mathrm{A}}^{\alpha}, && [X_{\mathrm{L}}^{\alpha},X_{\mathrm{S}}^{\beta}]=-\mathrm{i}\,\delta_{\alpha\beta}X_{\mathrm{S}}^{\tau},\\
&[X_{\mathrm{L}}^{\alpha},X_{\mathrm{S}}^{\tau}]=-\mathrm{i}X_{\mathrm{S}}^{\alpha}, && [X_{\mathrm{A}}^{\alpha},X_{\mathrm{S}}^{\beta}]=-\mathrm{i}\delta_{\alpha\beta}X_{\mathrm{A}}^{S},\\
&[X_{\mathrm{A}}^{\tau},X_{\mathrm{S}}^{\tau}]=\mathrm{i}X_{\mathrm{A}}^{S}, && [X_{\mathrm{S}}^{\alpha},X_{\mathrm{S}}^{\tau}]=-\mathrm{i}X_{\mathrm{L}}^{\alpha},\\
&[X_{\mathrm{S}}^{\alpha},X_{\mathrm{A}}^{S}]=-\mathrm{i}X_{\mathrm{A}}^{\alpha}, && [X_{\mathrm{S}}^{\tau},X_{\mathrm{A}}^{S}]=-\mathrm{i}X_{\mathrm{A}}^{\tau}.
\end{aligned} \tag{18}$$

These are isomorphic to the generators of the inhomogeneous pseudo-orthogonal group IO(4, 1) that keep the equation $\sum_{\alpha=1}^{4}(\mathrm{d}q^\alpha)^2-(\mathrm{d}q^5)^2=0$ invariant. The centre of the group consists only of the identity operator.

In the case of zero-mass particles, the corresponding Hamilton–Jacobi equation is

$$\Delta\equiv(S_\tau)^2-\sum_\alpha(S_\alpha)^2=0, \tag{19}$$

where S is Hamilton's action. The partial differential equations for ξ and φ are

$$\xi^\alpha_\alpha=\xi^\tau_\tau(\forall\alpha), \qquad \xi^\tau_\alpha=\xi^\alpha_\tau(\forall\alpha), \qquad \xi^\alpha_\beta+\xi^\beta_\alpha=0(\alpha\neq\beta), \qquad \varphi_\alpha=0=\varphi_\tau(\forall\alpha). \tag{20}$$

An analysis similar to that given above gives the following velocity vectors:

$$\begin{aligned}\xi^\tau&=a_4(S)+\sum_\beta b_{3+\beta}(S)q^\beta+b(S)\tau+C(S)(r^2+\tau^2)/2+\tau\sum_\beta C_\beta(S)q^\beta,\\ \xi^\alpha&=a_\alpha(S)+b(S)q^\alpha+\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_\gamma(S)q^\beta+b_{3+\alpha}(S)\tau-C_\alpha(S)(r^2-\tau^2)/2\\ &\qquad+q^\alpha\sum_\beta C_\beta(S)q^\beta+C(S)q^\alpha\tau,\\ \varphi(S)&=\sum_{n=0}^{\infty}d_nS^n/n!.\end{aligned} \tag{21}$$

The maximal symmetry group G has thus an infinite number of generators

$$\begin{aligned}&X^{S,n}=S^nX^S, \qquad X^{\alpha,n}=S^nX^\alpha, \qquad X^{\tau,n}=S^nX^\tau\\ &X_0^n=S^n\Big(\tau X^\tau+\sum_\alpha q^\alpha X^\alpha\Big), \qquad X_{\mathrm{R}}^{\alpha,n}=-\mathrm{i}S^n\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_\gamma q^\beta,\\ &X_{\mathrm{L}}^{\alpha,n}=S^n(\tau X^\alpha+q^\alpha X^\tau), \qquad X_{\mathrm{A}}^{\alpha,n}=S^n[-\tfrac12(r^2-\tau^2)X^\alpha+q^\alpha X_0^0],\\ &X_{\mathrm{A}}^{\tau,n}=S^n[\tfrac12(r^2-\tau^2)X^\tau+\tau X_0^0], \qquad n\geqslant0,\end{aligned} \tag{22}$$

with the non-vanishing commutators

$$\begin{aligned}&[X^{S,n},X^{\alpha,m}]=-\mathrm{i}mX^{\alpha,n+m-1}, \qquad [X^{S,n},X^{\tau,m}]=-\mathrm{i}mX^{\tau,n+m-1},\\ &[X^{S,n},X_{\mathrm{R}}^{\alpha,m}]=-\mathrm{i}mX_{\mathrm{R}}^{\alpha,n+m-1}, \qquad [X^{S,n},X_{\mathrm{L}}^{\alpha,m}]=-\mathrm{i}mX_{\mathrm{L}}^{\alpha,n+m-1},\\ &[X^{S,n},X_{\mathrm{A}}^{\alpha,m}]=-\mathrm{i}mX_{\mathrm{A}}^{\alpha,n+m-1}, \qquad [X^{S,n},X_{\mathrm{A}}^{\tau,m}]=-\mathrm{i}mX_{\mathrm{A}}^{\tau,n+m-1},\\ &[X^{\alpha,n},X_0^m]=-\mathrm{i}X^{\alpha,n+m}, \qquad [X^{\alpha,n},X_{\mathrm{R}}^{\beta,m}]=\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X^{\gamma,n+m},\\ &[X^{\alpha,n},X_{\mathrm{L}}^{\beta,m}]=-\mathrm{i}\,\delta_{\alpha\beta}X^{\tau,n+m}, \qquad [X^{\alpha,n},X_{\mathrm{A}}^{\beta,m}]=-\mathrm{i}\,\delta_{\alpha\beta}X_0^{n+m}+\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma,n+m},\\ &[X^{\alpha,n},X_{\mathrm{A}}^{\tau,m}]=-\mathrm{i}X_{\mathrm{L}}^{\alpha,n+m}, \qquad [X^{\tau,n},X_0^m]=-\mathrm{i}X^{\tau,n+m},\\ &[X^{t,n},X_{\mathrm{L}}^{\alpha,m}]=-\mathrm{i}X^{\alpha,n+m}, \qquad [X^{\tau,n},X_{\mathrm{A}}^{\alpha,m}]=-\mathrm{i}X_{\mathrm{L}}^{\alpha,n+m},\\ &[X^{\tau,n},X_{\mathrm{A}}^{\tau,m}]=-\mathrm{i}X_0^{n+m}, \qquad [X_0^n,X_{\mathrm{A}}^{\alpha,m}]=-\mathrm{i}X_{\mathrm{A}}^{\alpha,n+m}, \qquad [X_0^n,X_{\mathrm{A}}^{\tau,m}]=-\mathrm{i}X_{\mathrm{A}}^{\tau,n+m},\\ &[X_{\mathrm{R}}^{\alpha,n},X_{\mathrm{R}}^{\beta,m}]=\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma,n+m}, \qquad [X_{\mathrm{R}}^{\alpha,n},X_{\mathrm{L}}^{\beta,m}]=\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X_{\mathrm{L}}^{\gamma,n+m},\\ &[X_{\mathrm{R}}^{\alpha,n},X_{\mathrm{A}}^{\beta,m}]=\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X_{\mathrm{A}}^{\gamma,n+m}, \qquad [X_{\mathrm{L}}^{\alpha,n},X_{\mathrm{L}}^{\beta,m}]=-\mathrm{i}\sum_\gamma e_{\alpha\beta\gamma}X_{\mathrm{R}}^{\gamma,n+m},\\ &[X_{\mathrm{L}}^{\alpha,n},X_{\mathrm{A}}^{\beta,m}]=-\mathrm{i}\,\delta_{\alpha\beta}X_{\mathrm{A}}^{\tau,n+m}, \qquad [X_{\mathrm{L}}^{\alpha,n},X_{\mathrm{A}}^{\tau,m}]=-\mathrm{i}X_{\mathrm{A}}^{\alpha,n+m}.\end{aligned} \tag{23}$$

It is seen that the subgroup G_∞ with the generators for $n \geqslant 1$ is an invariant subgroup of G. The factor group $H \approx G/G_\infty$ consists of the generators with $n = 0$. H is the 15-parameter inhomogeneous pseudo-orthogonal group IO(3, 1) keeping the equation

$$\sum_{\alpha=1}^{3} (\mathrm{d}q^\alpha)^2 - (\mathrm{d}q^4)^2 = 0$$

invariant, together with the scaling transformation of the action. We can call this subgroup the physical symmetry group of the problem, since the other pieces of G can be mapped to H. This mapping, however, is not a homomorphism. The problem of whether the whole group G or the subgroup H is of real importance is not, of course, prejudged.

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